Problem :-

Calculate the length and orientation of a rod of length 5 metre in a frame of reference which is moving with velocity $0.6c$ in a direction making an angle 30° with the rod.

Sol:-

Given Proper light $l = 5$ metre. According to length contraction, the contraction takes trace only along to direction of motion; while perpendicular to direction of motion, no contraction taken place.

Component of length of rod along to direction of motion

$$l_x = l\cos 30^\circ = 5 \times \sqrt{3}/2 = 4.33\,m$$

Component of length of rod perpendicular to direction of motion

$$l_y = l\sin 30^\circ = 5 \times \frac{1}{2} = 2.5\,m$$

The component l_x, when viewed from moving frame

$$l'_x = l_x\sqrt{1 - \frac{v^2}{c^2}} = 4.33\sqrt{\left\{1 - \left(\frac{0.6c}{c}\right)^2\right\}}$$

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$$= 4.33 \times 0.8 = 3.46\,m$$

also $l'_y = l_y = 2.5\,m$

$\therefore$ Length of rod as observed from moving frame

$$l' = \sqrt{l'^2_x + l'^2_y}$$

$$= \sqrt{(3.46)^2 + (2.5)^2} = \sqrt{18.22} = 4.27\,m$$

If θ' is the angle made by rod with x-axis in moving frame, Then

$$\tan\theta' = \frac{l'_y}{l'_x} = \frac{2.5\,m}{3.46\,m} = \underline{0.7225}$$

$$\theta' = \tan^{-1}(0.7225) = 35.8^\circ.$$

Problem:- Show that the circle $x^2+y^2=a^2$ in frame S appears to be an ellipse in S' which is moving with velocity v relative to S along X-axis.

Sol:- Equation of circle in frame S is given as

$$x^2 + y^2 = a^2 \quad \text{—— (1)}$$

If (x', y') are coordinates in moving frame, Then According to length contraction

$$x' = x\sqrt{1-\frac{v^2}{c^2}}, \quad y' = y$$

$$\therefore \quad x = \frac{x'}{\sqrt{1-v^2/c^2}}, \quad y = y'$$

$$\therefore \quad \frac{x'^2}{1-\frac{v^2}{c^2}} + y'^2 = a^2$$

$$\Rightarrow \quad \frac{x'^2}{a^2\left(1-\frac{v^2}{c^2}\right)} + \frac{y'^2}{a^2} = 1$$

This is equation of ellipse. Hence a circle $x^2+y^2=a^2$ in frame S appears to be an ellipse in moving frame.

Show that $ds^2 = dx^2 + dy^2 + dz^2 - c^2 dt^2$ is invariant under Lorentz transformations.

Sol:- Lorentz transformations are

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}, \quad y' = y, \quad z' = z, \quad t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - c^2/v^2}}$$

$$\Rightarrow dx' = \frac{dx - v\,dt}{\sqrt{1 - v^2/c^2}}, \quad dy' = dy, \quad dz' = dz$$

$$dt' = \frac{dt - v/c^2\, dx}{\sqrt{1 - v^2/c^2}}$$

$$\therefore dx'^2 + dy'^2 + dz'^2 - c^2 dt'^2 = \frac{(dx - v\,dt)^2}{1 - v^2/c^2} + dy^2 + dz^2 - c^2 \left(\frac{dt - v/c^2\, dx}{\sqrt{1 - \frac{v^2}{c^2}}} \right)^2$$

$$\left\{ \frac{dx^2 + v^2 dt^2 - 2v\,dx\,dt}{1 - v^2/c^2} + dy^2 + dz^2 - c^2 \frac{\left(dt^2 + \frac{v^2}{c^4} dx^2 - \frac{2v}{c^2} dx\,dt \right)}{1 - v^2/c^2} \right\}$$

$$= \left\{ \frac{dx^2 (1 - v^2/c^2)}{\left(1 - \frac{v^2}{c^2}\right)} + dy^2 + dz^2 - \frac{c^2 dt^2 (1 - v^2/c^2)}{1 - \frac{v^2}{c^2}} \right\}$$

$$= dx^2 + dy^2 + dz^2 - c^2 dt^2 =$$ Lorentz invariant.

Prove that three dimensional volume element $dx\,dy\,dz$ is not invariant, but four dimensional volume element "$dx\,dy\,dz\,dt$" is invariant under Lorentz transformations.

Sol:- According to Lorentz Fitzgerald contraction, we have,

$$dx' = dx\sqrt{(1 - \beta^2)}$$
$$dy' = dy$$
$$dz' = dz$$

∴ ~~Thus~~

∴ Three dimensional volume element in system S'

$$= dx'\,dy'\,dz' = dx\sqrt{(1-\beta^2)}\cdot dy\cdot dz$$

$$= dx\,dy\,dz\cdot\sqrt{(1-\beta^2)}$$

$$\neq dx\,dy\,dz$$ (i.e., Three dimensional volume element in system S).

Thus three dimensional volume element is not invariant under Lorentz-transformations.

According to time dilation

$$dt' = \frac{dt}{\sqrt{(1-\beta^2)}}$$

∴ Four dimensional volume element in system

$$S' = dx'\,dy'\,dz'\,dt'$$

$$= dx\sqrt{(1-\beta^2)}\cdot dy\cdot dz\cdot\frac{dt}{\sqrt{(1-\beta^2)}}$$

$$= dx\,dy\,dz$$ (i.e., four dimensional volume element in system S).